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Buckling of Orthogonally Stiffened Finite Oval Cylindrical Shells under Axial Compression

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Presented herein is a theoretical investigation of the stability of an orthogonally stiffened, finite, oval cylindrical shell under axial compression for various types of boundary conditions. The mathematical model begins with the establishment of a set of suitable large deflection shell equations. The stiffeners are treated as classical beam elements having axial, bending, and twisting stiffnesses. The formulation includes prebuckling deformations and ring-stiffener discreteness. The problem is solved by employing a modal expansion together with the total energy expression. The modes of the expansion are generated from the numerical solution to the exact equilibrium equations and boundary conditions of a circular cylinder identical to the present one except for its constant radius of curvature which is equal to the average radius of the oval shell. The results for an infinitely long stiffened and unstiffened oval shell, under axial compression, agree very well with those of previous investigations. The effects of the types of support, the out-of-roundness of the oval, and the eccentricity of the stiffeners upon the stability of the oval cylinder are also presented.

I. Introduction

A MAJOR concern in the design of almost every aerospace vehicle is that it must be lightweight. In order to overcome this problem, the designer usually resorts to the use of thin-walled reinforced shells, because of their high structural efficiency.

The buckling of such structures is often the prime consideration in their design. Because of their prevalence in actual use, circular cylindrical shells have received considerable attention in the literature. Investigations on the buckling of axially compressed cylinders, such as the work of Donnell,[1] in trying to account for the disparity between the theoretical predictions and experimental observations, produced evidence that the effects of initial imperfections are very important. This topic has received continuing interest in the literature, a sample of which is given in survey papers.[2,3] Other important aspects of the stability problem are the influence of edge restraints[4,5] and the effects of prebuckling deformations.[6]

In regard to noncircular configurations, which are due either to special external shapes, internal storage requirements, or to imperfections arising unintentionally during manufacturing, only a relatively small amount of attention has been addressed to their study. The earliest investigation involving the stability of noncircular open cylinders was that of Marguerre.[7] More recently, Kempner, Chen and Feinstein performed analytical and experimental studies on the buckling of oval cylindrical shells under several types of loading.[8,9,23]

Achieving high structural efficiency in lightweight structures requires an understanding of the various effects of the stiffening members on the buckling of such structures. One of the most important effects is the eccentricity of stiffeners, as was first demonstrated by Van der Neut[10] and later elaborated by Singer.[11]

The purpose of the present study is to present an investigation of the stability of an orthogonally stiffened finite oval cylindrical shell under axial compression for various types of edge supports. The effects of both prebuckling deformations and ring discreteness are also included. The effects of discreteness are studied only for rings because in cylinders of practical size the longitudinal stiffeners (stringers) are usually closely spaced and therefore can be assumed to be averaged or "smeared out" over the stringer spacing.

The solution to the problem is found by applying the method of modal expansion. The idea here is to employ the solution of a simpler problem to solve a more complex one. This approach was highly refined by Young[12] as an extension of the Rayleigh-Ritz approach, and has been used recently for vibration studies of oval cylindrical shells by Boyd and Rao,[13] and Chen and Kempner.[14] In the present work, the buckling modes of an orthogonally stiffened finite circular cylinder are used to study the buckling of the corresponding cylinder with an oval cross section.

The preliminary problem of the buckling of a circular cylinder will be referred to as the equivalent circular cylinder problem. The required modal functions are developed by means of finite difference solutions of the exact buckling equations and boundary conditions which govern the buckling problem of the equivalent circular cylinder. The mode shapes describing the state for the lowest buckling load, together with those having the number of waves in the circumferential direction adjacent to the number of waves representing the critical state, will form the modal expansion for the displacement field of the oval cylinder.

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Upon application of the principle of stationary potential energy and utilization of the above mentioned expansion for the displacements, the stability investigation of the oval cylinder yields an eigenvalue problem which can be solved readily.

To test the present approach, the stability of an infinite oval cylinder with and without reinforcements was first investigated. The results compare very well with those obtained by other authors.[15,16] Subsequently, the influence of various sets of boundary conditions on the buckling behavior of a finite oval cylinder was examined. The results of the finite oval cylinder problem demonstrate the effects of the degree of out-of-roundness of the oval, the eccentricity of the stiffeners, and the types of supports upon the stability of the cylinder.

II. Basic Assumptions and Equations

The cylinder considered (Fig. 1) is composed of a shell stiffened by uniform, equally spaced rings and stringers. The shell and the stiffeners are homogeneous, isotropic, and linearly elastic, but the material is not necessarily the same for each of the components.

The coordinate system x, y, and z on the median surface of the shell, as shown in Fig. 1, is defined such that x is the axial coordinate, y is the circumferential coordinate, and z is the inward radial coordinate; L is the length of the shell of uniform thickness h and radius of curvature r, which is a function of the circumferential coordinate y. The subscripts r and s refer to quantities of the rings and stringers, respectively.

The geometry of the oval cylinder in the present work is characterized by its radius of curvature r which is represented by

$$r(y)=r_0/[1+\xi\cos(2y/r_0)] \tag{1}$$

where $0\leq|\xi|\leq 1$ is a measure of the eccentricity of the cross section of the cylinder, and r_0 is the radius of a circular cylinder having the same circumferential length L_0 of the oval; in other words, it is the mean radius of the doubly symmetric cross section $(r_0=L_0/2\pi)$.

It is assumed that the Kirchoff-Love hypotheses for thin shells are applicable. It is also assumed that curved beam theory, with twisting accounted for in an approximate manner, is applicable to the stiffeners. In cases where both rings and stringers lie on the same side of the shell, the effects of joints in the framework are ignored.

Within the framework of the preceding assumptions, and with the comma preceding the subscripts indicating partial differentiation with respect to the variable indicated by the subscript, the corresponding strain displacement relations for the thin shell appropriate for a stability analysis are (e.g., see Ref. 1):

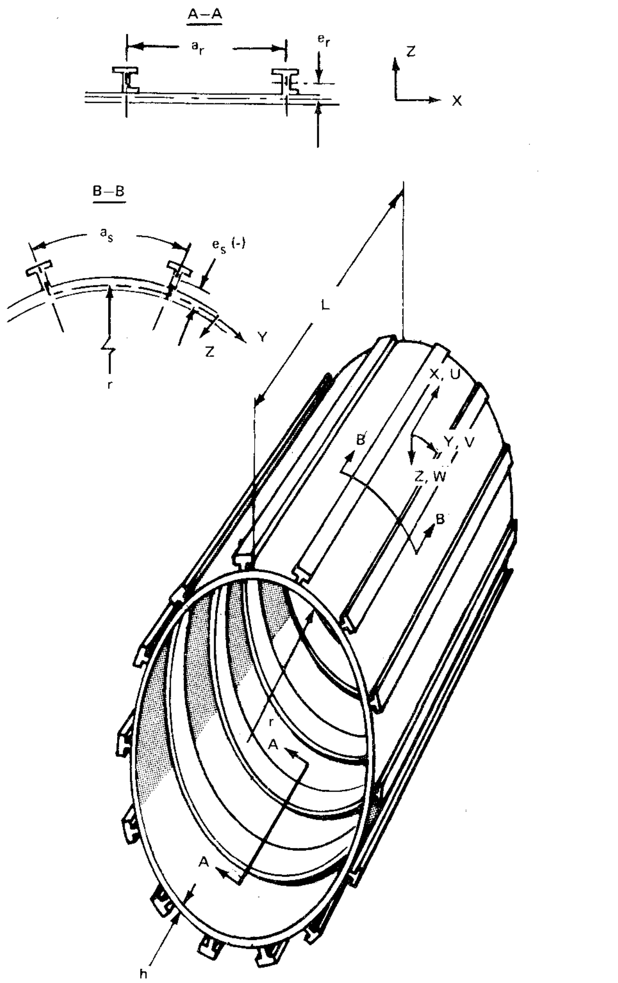
$$\epsilon_x=u_{,x}+\omega_y^2/2-zK_x;\ \ \epsilon_y=v_{,y}-w/r+\omega_x^2/2-zK_y$$

$$\gamma_{xy}=u_{,y}+v_{,x}-\omega_x\omega_y-2zK_{xy} \tag{2}$$

where u, v, and w denote the displacements of a point on the reference surface (see Fig. 2); ω_x and ω_y the rotations of the reference surface about the x and y axis, respectively; K_x and K_y are the changes in the curvature in the x and y directions, and K_{xy} the angle of twist. The explicit expressions for the rotations, curvature and twist associated with the theories of Donnell,[1] Love,[17] and Love-Reissner[18] are summarized in Ref. 19. For Donnell theory $\omega_x=w_{,y}$, $\omega_y=-w_{,x}$, $K_x=w_{,xx}$, $K_y=w_{,yy}$, and $K_{xy}=w_{,xy}$.

The stress components for the shells are expressed as

$$\sigma_x=E\frac{\epsilon_x+\nu\epsilon_y}{1-\nu^2};\ \ \sigma_y=E\frac{\epsilon_y+\nu\epsilon_x}{1-\nu^2};\ \ \tau_{xy}=\frac{E\gamma_{xy}}{2(1+\nu)} \tag{3}$$



QUANTITY			UNSTIFFENED CYLINDER	STIFFENED CYLINDER
R_0	cm	(in.)	127 (50)	508 (200)
L	cm	(in.)	89 (35)	508 (200)
h	cm	(in.)	1.27 (0.5)	0.25 (0.1)
ν		—	0.30	0.33

Fig. 1 Geometry of stiffened oval cylinder.

In regard to stiffeners, the strains and stresses are of the form

$$\epsilon_{x_s}=\epsilon_x \qquad \epsilon_{y_r}=\epsilon_y \tag{4}$$

and

$$\sigma_{x_s}=E_s\epsilon_x \qquad \sigma_{y_r}=E_r\epsilon_y \tag{5}$$

Here the E's and ν are the moduli of elasticity and the Poisson's ratio of the materials composing the stiffened cylinder, respectively. Furthermore, it is assumed that the stiffeners twist in such a fashion that their angles of twist, $\theta_s(=\omega_{x,x})$ and $\theta_r(=-\omega_{y,y})$ are equal to the appropriate angles of rotations of the shell at their lines of contact.

The resultant middle surface forces and moments of the isotropic shell are defined as follows (see Fig. 2):

$$(N_x,N_y,N_{xy},M_x,M_y,M_{yx},M_{xy})=\int_{-h/2}^{h/2}[\sigma_x+\sigma_{x_s},\sigma_y+\sigma_{y_r},\tau_{xy}$$

$$z(\sigma_x+\sigma_{x_s}),z(\sigma_y+\sigma_{y_r}),z\tau_{xy},-z\tau_{xy}]\mathrm{d}z \tag{6}$$

It should be pointed out that these expressions reflect the fact tha the shell thickness is small compared to the radius and therefore, differ from those of the more elaborate format of Flügge.[20]

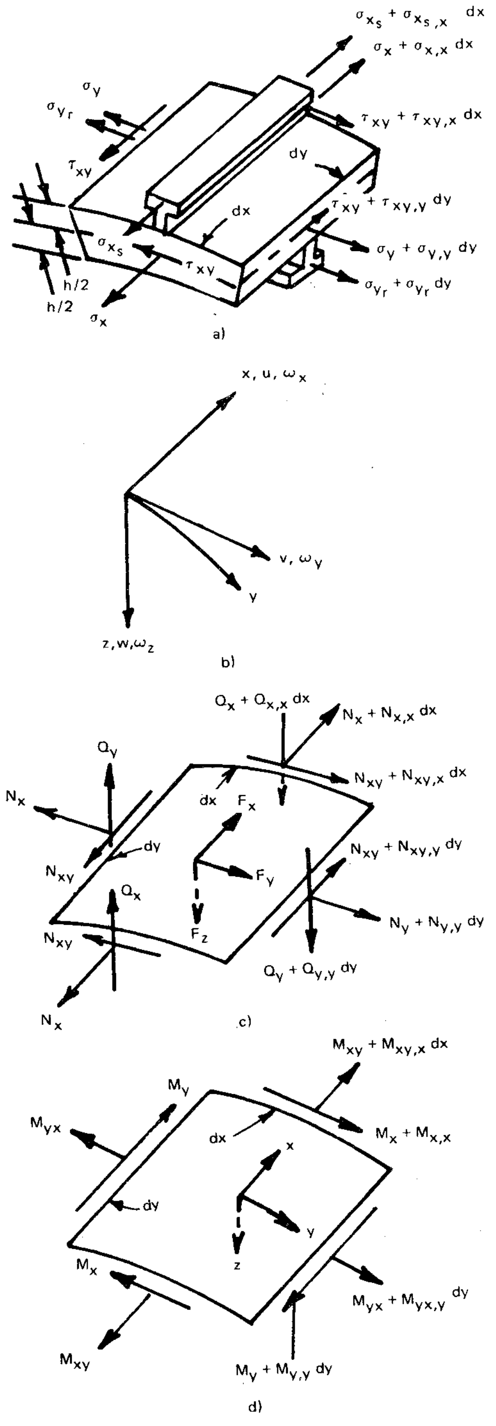


Fig. 2 Conventions for coordinates, displacements, rotations stresses, and stress resultants.

III. Development of the Potential Energy

The conditions of equilibrium for classical buckling problems can be obtained from the following variation of the total potential energy Π_T

$$\delta\Pi_T = \sum_{m=1}^{3} \delta U_m + \delta V_L \tag{7}$$

where

$$\delta U_1 = \int_V (\sigma_x \delta\epsilon_x + \sigma_y \delta\epsilon_y + \tau_{xy}\delta\gamma_{xy}) \mathrm{d}V$$

$$\delta U_2 = \sum_{i=1}^{N_1} \left[\int_{V_r} \sigma_{y_r}\delta\epsilon_{y_r} \mathrm{d}V_r + \int_{y_r} G_r J_r \theta_r \delta\theta_r \mathrm{d}y_r \right]_i$$

$$\delta U_3 = \sum_{j=1}^{N_2} \left[\int_{V_s} \sigma_{x_s}\delta\epsilon_{x_s} \mathrm{d}V_s + \int_{x} G_s J_s \theta_s \delta\theta_s \mathrm{d}x \right]_j \tag{8}$$

are the changes in strain energy stored in the shell, the ring, and the stringers, respectively. The second term in each of the last two expressions represents the approximate form of the twisting energy of the stiffeners. This term follows directly from the classical formulation of Saint-Venant.[17] The quantities J_r, J_s, N_1, and N_2 are the torsional constants of the cross sections and the number of discrete rings and stringers, respectively. The symbols $\mathrm{d}V$, $\mathrm{d}V_r$, and $\mathrm{d}V_s$ are the volume elements of the shell, rings and stringers, respectively; and $\mathrm{d}y_r$ is the length element measured along the line through the centroid of the rings. The variations of the angles of twist are $\delta\theta_r$ and $\delta\theta_s$.

The term δV_L is the potential energy of the external loads and is given by:

$$\delta V_L = -\int_A [F_x\delta u + F_y \delta v + F_z \delta w] \mathrm{d}A \tag{9}$$

where F_x, F_y, and F_z are the physical external loads applied to the undeformed reference surface (see Fig. 2); δu, δv, and δv, and δw are the independent variations of the displacement field; and $\mathrm{d}A$ is the element of the cross-sectional area of the shell wall. For the present problem, retaining first order terms only, Eq. (9) can be taken as

$$\delta V_L = -\int_A q\delta w \mathrm{d}A + \int_y [\hat{N}_x(\delta u - \bar{e}\delta w_{,x})] \Big|_{x=0}^{x=L} \mathrm{d}y \tag{10}$$

in which q is the lateral pressure, $\hat{N}_x$ the axial compressive force, and $\bar{e}$ the distance from the middle surface of the shell to the line on which $\hat{N}_x$ acts. This quantity is introduced so that the loading of edge moments and axial compression may be represented by a statically equivalent force system.

In order to obtain the stability conditions from the variational relations, the principle of stationary potential energy will be invoked, with the reinforced shell considered to be in a state of neutral equilibrium. Consequently, two adjacent equilibrium positions can be assumed to exist.

The first state is characterized by the initial prebuckling displacements u^0, v^0, and w^0 measured from the undeformed median surface, while the second corresponds to the buckling displacement components.[21] Thus,

$$u = u^0 + \alpha u' \quad v = v^0 + \alpha v' \quad w = w^0 + \alpha w' \tag{11}$$

Here the prime refers to the additional (buckling) displacements measured from the initial deformed state, and α represents an infinitesimal quantity independent of the geometrical coordinates. Introducing the preceding equations into the strains, stresses, and strain energy relations, the resulting expressions can be cast in the form $(\ ) = (\ )^0 + \alpha(\ )' + \alpha^2(\ )''$ where the symbol $(\ )''$ denotes the coefficient of the α^2 terms. Since the principle of stationary potential energy states that a necessary condition of the equilibrium of any given state is that the variation of the total potential vanishes, it follows that

$$\delta\Pi_T = \delta\Pi_T^0 + \delta\Pi_T' + \delta\Pi_T'' = 0 \tag{12}$$

Since the prebuckling displacements correspond to an equilibrium state, only variations of the additional displacements need be undertaken. Hence, $\delta\Pi_T^0 = 0$. Furthermore, for the equilibrium (neutral or otherwise) of the prebuckling state, the principle of virtual work states that

$\delta\Pi_T''=0$. Therefore, the condition characterizing stability, for the two states to be in equilibrium, reduces to $\delta\Pi_T''=0$. In explicit form, this equation can be written as

$$\delta\Pi_T''=\int_V [\sigma_x'\delta\epsilon_x'+\sigma_y'\delta\epsilon_y'+\tau_{xy}'\delta\gamma_{xy}'+\sigma_x^0\delta\epsilon_x''+\sigma_y^0\delta\epsilon_y''$$

$$+\tau_{xy}^0\delta\gamma_{xy}'']\mathrm{d}V+\sum_{i=1}^{N_1}\left\{\int_{V_r}(\sigma_{y_r}'\delta\epsilon_{y_r}'+\sigma_{y_r}^0\delta\epsilon_{y_r}'')\mathrm{d}V_r\right.$$

$$\left.+\int_{y_r} G_rJ_r(\theta_r'\delta\theta_r'+\theta_r^0\delta\theta_r'')\mathrm{d}y_r\right\}_i$$

$$+\sum_{j=1}^{N_2}\left\{\int_{V_s}(\sigma_{x_s}'\delta\epsilon_{x_s}'+\sigma_{x_s}^0\delta\epsilon_{x_s}'')\mathrm{d}V_s\right.$$

$$\left.+\int_x G_sJ_s(\theta_s'\gamma\theta_s'+\theta_s^0\delta\theta_s'')\mathrm{d}x\right\}_j=0 \tag{13}$$

The applied loads $\hat{N}_x$ and q will appear in the superscript zero terms.

IV. Equilibrium Equations for Oval Cylinders

The equilibrium equations and appropriate boundary conditions are obtained by applying to Eq. (13) the standard variational procedure based on the three independent variations δu, δv, and δw. Thus, for Donnell's accuracy, the following combined equations of equilibrium result:

$$N_{x,x}+N_{xy,y}=0;\quad N_{y,y}+N_{xy,x}=0$$

$$M_{x,xx}+M_{y,yy}+M_{yx,xy}+N_y/r+N_xw_{,xx}$$

$$+N_yw_{,yy}+2N_{yx}w_{,xy}+N_{y,y}w_{,y}+q=0 \tag{14}$$

The corresponding sets of homogeneous boundary conditions along the edges $x=0$ and $x=L$ are found to be

u prescribed or N_x prescribed

v prescribed or N_{xy} prescribed

$w_{,x}$ prescribed or $M_x+\hat{N}_x\bar{e}$ prescribed

w prescribed or $(M_{x,x}+N_xw_{,x}+N_{xy}w_{,y}$

$-M_{xy,y})$ prescribed (15)

Due to the closed configuration of the shell in the circumferential direction, these are the only boundary conditions needed. The explicit expressions for the resultant forces and moments, as well as equilibrium equations and boundary conditions corresponding to the aforementioned Love and Love-Reissner theories are given in Ref. 19.

V. Modal Expansion

As stated earlier, the condition characterizing elastic stability (neutral equilibrium) is given by Eqs. (13). The displacement field u, v, and w, which describes the deformed state of the cylinder, will be expressed in terms of modal functions which characterize the buckling states of a reinforced cylinder identical in every respect to the present one except that its radius of curvature is constant, i.e., it has a circular cross section of radius r_0. The expressions for the displacement field, given by Eq. (11), in terms of these expansions are written as:

$$u=u^0+\alpha\sum_{m=1}^{\infty}\gamma_mX_mU_n(x)\cos(ny/r_0)$$

$$v=v^0+\alpha\sum_{m=1}^{\infty}\gamma_mY_mV_n(x)\sin(ny/r_0)$$

$$w=w^0+\alpha\sum_{m=1}^{\infty}\gamma_mZ_mW_n(x)\cos(ny/r_0) \tag{16}$$

where U_n, V_n, and W_n are the modal functions. The term γ_m is equal to $1/\sqrt{2}$ if $n=0$ and is equal to 1 otherwise. The quantities X_m, Y_m, and Z_m form the unknown eigenvectors that need to be found. In the above expressions, $m=[(n+2)/2]$ if n is even and $m=[(n+1)/2]$ if n is odd. The prebuckling displacement field for the stiffened oval cylinder is obtained in a similar fashion to the relatively simple approximate solution in Ref. 22 and later used in Ref. 23. Such approximations give excellent agreement between the simplified and the exact solution of the linearized equations for both clamped and simply supported cylinders.[22,24]

The modal functions U_n, V_n, and W_n appearing in Eqs. (16) are, in fact, the normalized buckling solution of the equivalent finite circular cylinder. The detailed procedure which leads to the determination of these functions may be found in Ref. 19.

In the manner outlined in Ref. 22 and 23, the prebuckling problem of the oval cylinder is solved by applying the equilibrium equations (14) and the boundary conditions (15) for the prebuckling state, i.e., retaining only the axisymmetric terms in Eqs. (14) yields:

$$N_{x,x}^0=0;\quad M_{x,xx}^0+N_y^0/r_0+N_x^0w_{,xx}^0=-q \tag{17}$$

where $(\ )^0$ denotes prebuckling quantities and r_0 has been substituted for r. The first of Eqs. (17) states that N_x^0 is a constant, which in the present problem must be equal to the negative of the applied axial compressive load $\hat{N}_x$. By smearing out the stringers and by integration of the first of Eqs. (17), and by expressing the resultant forces and moments in terms of displacements, it follows that

$$K(u_{,x}^0+w_{,x}^{0^2}/2-\nu w^0/r_0)+(E_sA_s/a_s)(u_{,x}^0+w_{,x}^{0^2}/2$$

$$-e_sw_{,xx}^0)(1-e_s/r_0)=-\hat{N}_x \tag{18}$$

where $K=Eh/(1-\nu^2)$. By introducing Eq. (22) into the second of Eqs. (17) it can be shown that

$$-\Omega_1w_{,xxxx}^0-\Omega_2w_{,xx}^0-\left[\Omega_3+\sum_{i=1}^{N_1}\Omega_4\delta(x-ia_1)\right]w^0+\Omega_5=0 \tag{19}$$

where the quantities Ω_1 to Ω_5 are a collection of terms containing the material and geometric parameters; they can be found in Appendix A of Ref. 19. A closed form solution to the above equation, for both clamped and simply supported edges, with the rings smeared out, is given in Appendix B of Ref. 26. In this solution it is observed that the prebuckling displacement w^0 appears with terms $\hat{N}_xr_0$ and qr_0^2 in addition to terms describing the material properties and the geometry excluding r_0. Following the line of reasoning given in Ref. 23, the prebuckling displacement w^0 for the oval cylinder can be expressed in terms of the applied loads and a function w_c^0 as

$$w^0=(\lambda R+\psi R^2)w_c^0 \tag{20}$$

where $R=r/r_0$, $\lambda=\nu\bar{K}\hat{N}_x/(\nu\bar{K}\hat{N}_{x_c}-q_cr_0)$ and $\psi=qr_0/(q_cr_0-\nu\bar{K}\hat{N}_{x_c})$, respectively, are the ratios of the local radius of curvature, the axial compressive load parameter, and the lateral pressure parameter of the oval cylinder to the "equivalent" load of the circular cylinder; and $\bar{K}=K/[K+E_sA_s(1-e_s/r_0)/a_s]$. The term w_c^0 is the radial deflection of the equivalent circular cylinder. For the present problem this deflection is computed numerically at locations along the length of the cylinder. Due to the presence of the discrete rings, which represent discontinuities in the cylinder, no closed form solution for w^0 can be obtained.

Equation (20) indicates that to find the prebuckling deflection of the oval shell under axial compression ($\psi=0$), or lateral pressure ($\lambda=0$), or a combination of both, with $\hat{N}_x$ or q prescribed, i.e., $\lambda \neq 0$ or $\psi \neq 0$, the equivalent circular cylinder problem should first be solved and then the deflection scaled according to the ratio of the buckling load of the oval cylinder to that of the circular cylinder. In general, the ratio of the buckling loads of the oval and circular cylinders is less than unity. The term u^0 can be found from Eq. (18).

In order to arrive at the final form of the condition characterizing buckling of the stiffened oval cylindrical shell, the function $r(y)$ given in Eq. (1) is first expanded in Fourier series and then the resulting expression, together with Eqs. (16) and (20), is substituted into Eq. (13). Thus, with the aid of the orthogonal properties of trigonometric functions, the second variation of the total potential energy, Eq. (18), can be written as

$$\delta\Pi_T'' = ([A]+\xi[B]+\xi^2[C]) \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} + \lambda[D] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix}$$

$$+\psi[E] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} + (\lambda^2+\psi^2)[F] \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = 0 \qquad (21)$$

Here the terms A to F are $3N \times 3N$ (N is the number of terms in the modal expansion) symmetric matrices whose components are given in Ref. 19; and X, Y, and Z are the three components of the $3N$ dimensional eigenvector; λ and ψ are the unknown eigenvalues to be established. The last term is the contribution of the square of the prebuckling displacement w^0. Even though such terms are often neglected in similar situations, they are retained in the present study.

VI. Method of Solution

Once the geometry, edge, and load conditions are specified, the first items to be computed are the modal functions $U_n(x)$, $V_n(x)$, and $W_n(x)$, which are generated by solving the equivalent circular cylinder problem.

The choice of these functions and the number of terms is determined as follows. First the critical load of the equivalent circular cylinder is found. This quantity is required to determine the prebuckling state of the oval cylinder, given by Eq. (20), and is also an indication of the buckling load of the oval. In Ref. 15 it has been shown that for the buckling of an infinite, unstiffened oval shell under axial compression, an excellent first approximation is obtained by substituting the maximum radius of curvature into the well known expression for circular cylinders, i.e., for $\nu=0.3$

$$\tilde{N}_{x_c} = (0.6Eh^2/r_0) \qquad (22)$$

and thus,

$$\lambda_{\text{approx}} = (\tilde{N}_x/\tilde{N}_{x_c}) = (r_0/r_{\max}) = 1 - |\xi| \qquad (23)$$

Next, with the critical load of the circular cylinder and the corresponding mode shape known, i.e., n_{crit} known, a set of consecutively numbered modes bracketing the mode with the critical value n_{crit} is selected for the expansion in Eq. (16). Due to the orthogonality conditions of trigonometric functions, the odd and even terms of the above mentioned expansion are uncoupled, and hence, they are considered separately. In many cases, these modes are computed while searching for the critical load. A total number of nine modes was sufficient in most cases.

The independent modal functions $U_n(x)$, $V_n(x)$, and $W_n(x)$ are obtained by normalizing the buckling mode shapes of the equivalent circular cylinder with respect to the maximum value of each displacement component. The coefficients X_m, Y_m, Z_m are also independent.

After the modal functions and the prebuckling displacement w^0 are known, the eigenvalue problem given by Eq. (21) is solved using the Stodola-Vianello method.[25] This procedure, which is also known as the power or iterative method, will give the lowest eigenvalue (λ or ψ) and the corresponding eigenvectors (X_n, Y_n, and Z_n). To apply Stodola's method, Eq. (21) must first be rearranged. With λ or ψ known, Eq. (21) can be written as

$$[\tilde{A}]\{\tilde{x}\} + \theta[\tilde{B}]\{\tilde{x}\} + \theta^2[\tilde{C}]\{\tilde{x}\} = 0 \qquad (24)$$

where $[\tilde{A}]$, $[\tilde{B}]$, and $[\tilde{C}]$ are coefficients matrices, θ denotes the unknown eigenvalue (buckling load), and $\{\tilde{x}\}$ is the corresponding eigenvector.

Since the lowest eigenvalue is required, Eq. (24) is dealt with in the form

$$([\tilde{A}]^{-1}[\tilde{B}] + (1/\theta)[I] + \theta[\tilde{A}]^{-1}[\tilde{C}])\{\tilde{x}\} = 0 \qquad (25)$$

Here $[I]$ represents the identity matrix. With the exception of the last term, the above equation is in the appropriate form for Stodola's iteration method, which solves for the largest value of $1/\theta$ (smallest θ). To initiate a solution of Eq. (25), the last term can be treated as a known quantity and can be added to the first term; e.g., the last term can be assumed to be zero or it can be approximated by Eq. (23). The result is a linearized form of Eq. (25) which can be solved with less difficulty. An even better approach is to substitute the eigenvalue obtained by using one of the two above approximations in the last term, and to re-evaluate the eigenvalues. As it will be shown in the next section, the present study considers and compares all of the above approaches.

For cases where the boundary conditions to be satisfied are of the displacement type, i.e, $u=v=w=w_{,x}=0$, the modal functions generated in the equivalent circular problem will satisfy those for the oval cylinder. In cases where the edge conditions are on stress resultants, i.e., $N_x = M_x = N_{xy} = 0$, it is found necessary to impose additional constraint conditions on the eigenvalue problem. These conditions arise from the normalization of the mode shapes of the circular cylinders.

For example, the case where $v=w=N_x=M_x=0$ on the boundary implies that at the edges of the circular cylinder $v_c=w_c=0$. Hence, the corresponding modal functions will also be zero, i.e., $V_n = W_n = 0$. This in turn results in $v=w=0$ at the edges of the oval cylinder. The conditions on the stress resultant lead to the following for both the circular and oval cylinders:

$$w'_{,xx} = 0 \quad u'_{,x} + w'_{,x} w^0_{,x} = 0 \qquad (26)$$

For the circular cylinder problem, Eqs. (26) yield $w_n^{\bullet\bullet} = U_n^{\bullet} + W_n^{\bullet} w^0_{c,x}) = 0$, while for the oval problem, the results are:

$$W_n^{\bullet\bullet} = 0, \quad X_n = Z_n(-W_n^{\bullet} w^0_{,x}/U_n^{\bullet}) \qquad (27)$$

In the above $(\ )^{\bullet} = d(\ )/dx$. Therefore, in order to satisfy the above boundary conditions the second of Eqs. (27) must be introduced as a constraint condition on the modal shape. This condition can be inserted into Eq. (25) thereby reducing the problem to sets of unknown eigenvectors Y_n and Z_n.

Other boundary conditions can also be applied following this approach. In this study three types of boundary conditions were examined; by employing the well known terminology of Ref. 4, they can be classified as:

$$\text{C1:}\quad u=v=w=w_{,x}=0$$

$$\text{S2:}\quad v=w=N_x=M_x=0$$

$$\text{S4:}\quad w=N_x=N_{xy}=M_x=0 \tag{28}$$

The choice of these cases is due to the fact that they are also representative of the other five types. The C1 case does not require constraint conditions, whereas the S2 (discussed in the above paragraph) and the S4 conditions do require them.

VII. Discussions and Conclusions

The geometries of the cylinders studied are given in Figs. 1 and 3. For such cylinders, the shell theory considered herein has been found to be very accurate. A comparison of the accuracy of the Donnell, Love and Love-Reissner theories for the buckling of the externally stiffened circular cylinder under axial compression, given in Ref. 19, shows excellent agreement of the results. This suggests that the Donnell theory would be the best choice, because it offers both simplicity and sufficient accuracy. Even though this comparison was carried out for circular cylinders, it is believed that the results for the oval cylinder will be similar, since the eigenfunction expansion depends upon the results of the circular cylinder. As a consequence, the Donnell theory is adopted for all subsequent calculations in this study.

The results of the lowest buckling load, and the corresponding number of waves along the circumference of the equivalent circular cylinder problems investigated are given in Table 1. These results have been found to be in excellent agreement with those published in Refs. 4 and 26 which give results for a number of geometries. It should be pointed out that the authors are not aware of any previous results for cases C1 and S4 for the stiffened cylinders. The results for these two cases show a similar trend to those of the unstiffened cylinder.

For the problems which have the smallest buckling load corresponding to less than ten waves along the circumference, i.e., $n<10$, the number of terms required is about 18. However, since the even and odd modes are uncoupled, the actual number of terms used in the calculations is ony half, i.e., 9. This was found to be generally valid throughout this study, except for the case of the externally stiffened cylinder under axial compression for which $n=114$. The truncation of the series (modal) expansion on the lower side is usually limited by $n=0$ for even modes, or $n=1$ for odd modes; while, on the upper side, the truncation is determined by the n corresponding to a converging value of the buckling load or a load that is greater than the minimum value in the series, e.g., $2\times$ buckling load. The sufficiency of the number of terms is verified by observing the eigenvector of the oval problem.

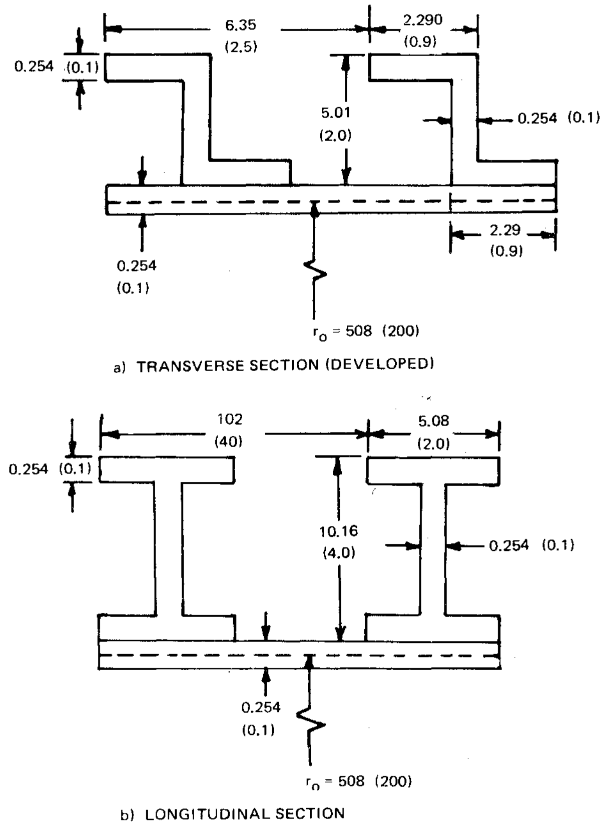


Fig. 3 Dimensions of stiffened oval cylinder. Dimensions are in cm (in.). $L=r_0=508$ cm (200 in.).

The variation of the buckling load determined for an infinitely long, unstiffened and stiffened (internally and externally) oval cylinder under axial compression, i.e., the classical case, is shown in Table 1. The agreement of the results with those of Refs. 15 and 16 is evident. This problem was solved by using sinusoidal functions for the modal shapes. The differences of the results of the stiffened cylinder is believed to be due to the greater number of rings considered in Ref. 16; that is, 10 vs 4.

The results for the unstiffened and externally stiffened finite oval cylinder are given in Tables 2 an 3. These tables

Table 1 Results of infinitely long, oval, unstiffened, and stiffened cylinder under axial compression

	Unstiffened cylinder, $(R_0/h)=100$ $\lambda=(\hat{N}_x/\hat{N}_{xc})$ [a]		Stiffened cylinder, $(R_0/h)=2000$			
			Present theory $\lambda=(\hat{N}_x/\hat{N}_{xc})$ [b]		Ref. 16 $\lambda=(\hat{N}_x/\hat{N}_{xc})$ [c]	
ξ	Present theory	Ref. 15	Internally stiffened	Externally stiffened	Internally stiffened	Externally stiffened
0	1.0	1.0	0.6513	1.0	0.6635	1.0
−0.1	0.918	0.910	0.6339	0.9440	...	...
−0.2	0.817	0.816	0.6037	0.8596	0.6172	0.8388
−0.3	0.725	0.720	0.5560	0.7606	...	...
−0.4	0.630	0.624	0.5046	0.6560	0.5154	0.6470
−0.5	0.530	0.520	0.4459	0.5530	...	...
−0.6	0.445	0.440	0.3817	0.4530	0.3842	0.4631
−0.7	0.345	0.340	0.3128	0.3679	...	...
−0.8	0.255	0.250	0.2440	0.2706	0.2483	0.2780
−0.9	0.172	0.170	0.1661	0.1853	...	...
−1.0	0.105	0.101	0.1035	0.1202	0.1170	0.1400

[a]These values are nondimensionalized with respect to 0.00603, which corresponds to the buckling of an infinitely long unstiffened cylinder. [b]These values are nondimensionalized with respect to 0.0109, which corresponds to the buckling load of an externally reinforced circular cylinder with 4 rings. [c]These values are nondimensionalized with respect to 0.01252, which corresponds to the buckling load of an externally reinforced circular cylinder with 10 rings.

Table 2 Buckling load of finite, unstiffened, oval cylinder under axial compression

	C1 case[a] $N_{xc}=5.5059\times10^{-3}\,Eh$			S2 case $\hat{N}_{xc}=4.88\times10^{-3}\,Eh$					S4 case $\hat{N}_{xc}=3.0326\times10^{-3}\,Eh$			
ξ	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_4	λ_5	λ_1	λ_2	λ_4	λ_5
0	0.969	1.001	1.001	0.971	1.001	1.001	0.971	0.9998	0.976	1.039	0.938	0.999
−0.1	0.927	0.959	0.961	0.939	0.969	0.970	0.956	0.956	0.864	0.926	0.894	0.951
−0.2	0.863	0.895	0.899	0.875	0.905	0.906	0.898	0.898	0.792	0.851	0.865	0.917
−0.3	0.793	0.827	0.833	0.812	0.842	0.847	0.839	0.839	0.722	0.782	0.839	0.886
−0.4	0.726	0.757	0.769	0.755	0.783	0.792	0.784	0.784	0.658	0.720	0.813	0.856
−0.5	0.663	0.695	0.708	0.696	0.734	0.748	0.739	0.772	0.601	0.666	0.783	0.819
−0.6	0.603	0.610	0.653	0.621	0.681	0.716	0.699	0.740	0.549	0.617	0.751	0.782
−0.7	0.546	0.579	0.614	0.543	0.599	0.665	0.649	0.725	0.500	0.573	0.715	0.740
−0.8	0.487	0.523	0.589	0.452	0.501	0.590	0.605	0.736	0.443	0.520	0.676	0.695
−0.9	0.394	0.429	0.554	0.324	0.359	…	0.558	0.752	0.346	0.417	0.634	0.648

$\lambda_{1,2,3,4,5}=\hat{N}_x/\hat{N}_{xc}$. λ_1—Omits λ^2 terms; omits constraints. λ_2—Includes λ^2 terms by using $(1-|\xi|)$ approximation; omits constraints. λ_3—Includes λ^2 terms by using λ_2 values; omits constraints. λ_4—Omits λ^2 terms; includes constraints. λ_5—Includes λ^2 terms by using $(1-|\xi|)$ approximation; includes constraints.

[a]Case C1 does not require constraints.

Table 3 Buckling load of finite, externally stiffened, oval cylinder under axial compression

	C1 case[a] $\hat{N}_{xc}=1.0552\times10^{-2}\,Eh$			S2 case $\hat{N}_{xc}=8.642\times10^{-3}\,Eh$				S4 case $\hat{N}_{xc}=6.558\times10^{-3}\,Eh$		
ξ	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3	λ_4	λ_5	λ_1	λ_2
0	0.986	1.0004	1.0004	0.821	1.00001	1.00001	0.809	0.997	0.755	1.021
−0.1	0.946	0.961	0.961	0.798	0.976	0.991	0.808	1.009	0.670	0.916
−0.2	0.891	0.906	0.910	0.754	0.926	0.956	0.802	1.007	0.597	0.830
−0.3	0.838	0.852	0.856	0.705	0.783	0.921	0.791	1.011	0.527	0.747
−0.4	0.786	0.802	0.807	0.652	0.816	0.886	0.757	1.016	0.462	0.666
−0.5	0.739	0.755	0.763	0.588	0.715	0.788	0.820	1.038	0.400	0.584
−0.6	0.697	0.714	0.727	0.501	0.605	0.666	0.716	1.091	0.339	0.505
−0.7	0.657	0.676	0.701	0.401	0.488	0.545	0.701	1.219	0.278	0.418
−0.8	0.616	0.639	0.694	0.311	0.364	0.415	0.698	1.250	0.212	0.319
−0.9	0.548	0.578	0.756	0.195	0.225	…	…	…	0.134	0.198

$\lambda_{1,2,3,4,5}=\hat{N}_x/\hat{N}_{xc}$. λ_1—Omits λ^2 terms; omits constraints. λ_2—Includes λ^2 terms by using $\{1-|\xi|\}$ approximation; omits constraints. λ_3—Includes λ^2 terms by using λ_2 values; omits constraints. λ_4—Omits λ^2 terms; includes constraints. λ_5—Includes λ^2 terms by using $\{1-|\xi|\}$ approximation; includes constraints.

[a]C1 case does not require constraints.

present the results for Eq. (25) by employing the three approaches suggested in Sec. VII. In addition, cases S2 and S4 also account for the constraint conditions [i.e., see Eq. (27)] required for these cases. The only exception is case S4 for the stiffened problem. In trying to include the constraint conditions in the S4 case of the stiffened cylinder problem, the expressions for the forces and moments to be satisfied on the boundary yield a system of equations that have only the trivial solution $X_n=Y_n=Z_n=0$ due to the presence of the $u_{,x}$ term contribution arising from the stiffeners. This is an indication that the problem is over-constrained and, consequently, the constraint conditions are omitted. This means that the boundary conditions are not satisfied exactly. The corresponding cases of the unstiffened cylinder do not encounter these problems because the system of equations arising from the constraints has a non-trivial solution.

For the C1 clamped cylinder cases (Tables 2 and 3) the results from all three approaches for $|\xi|\leq0.5$ are all within 6% of each other. For $|\xi|>0.5$ the differences increase. The reason for this is that for $|\xi|>0.5$ the approximation for the prebuckling displacement given by Eq. (20) and Fig. 4 introduces some errors. Hence, the contribution of the prebuckling deformations is magnified. It is believed that either the linearized approximation, i.e., neglect λ^2 terms, or the engineering approximation approach can be used with confidence. This is because, if one compares the reduction of the eigenvalues for $|\xi|>0$ with respect to the eigenvalue for $\xi=0$, the difference in the results using the two approaches (linearized and engineering approximation) are even smaller.

In Fig. 5, the results for the clamped, unstiffened cylinder using the linearized approach are compared to the theoretical results of Ref. 23 and to the classical case, Table 1. As the curves show, the longer cylinder considered in Ref. 23 ($L/r=5$) behaves more like the classical cylinder than the shorter cylinder ($L/r=0.7$) considered in the present study.

The results of the simply supported cases S2 and S4 indicate that for small values of ξ, $|\xi|<0.3$, the constraint conditions offer little improvement over the results for the cases when they are omitted. If anything, for $|\xi|>0$ they introduce some errors due to the fact that the average value along the circumference was used for the prebuckling deformation appearing in the constraint conditions, i.e., Eq. (27). As ξ increases, the error is even larger. Therefore, the constraint conditions can be omitted from the analysis without any additional penalties than those already introduced by the approximation for the prebuckling displacements.

For cases S2 and S4 of the unstiffened cylinder (Table 2) either the linearized approximation λ_1 (which omits λ^2 terms) or the engineering approximation for the nonlinear terms yielding λ_3 can be used with confidence, because, in terms of reduction from the value of the load for the circular cylinder, the results are close. However, for the S2 and S4 cases of the externally stiffened problem (Table 3) the engineering approximation corresponding to λ_3 is recommended due to the

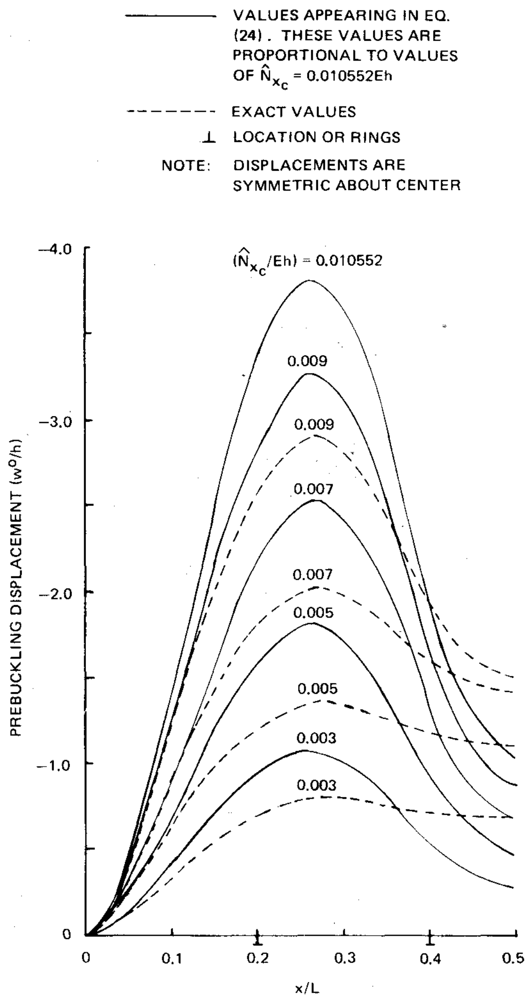


Fig. 4 Comparison of exact and approximated values of prebuckling displacements (C1 case, externally stiffened circular cylinder under axial compression).

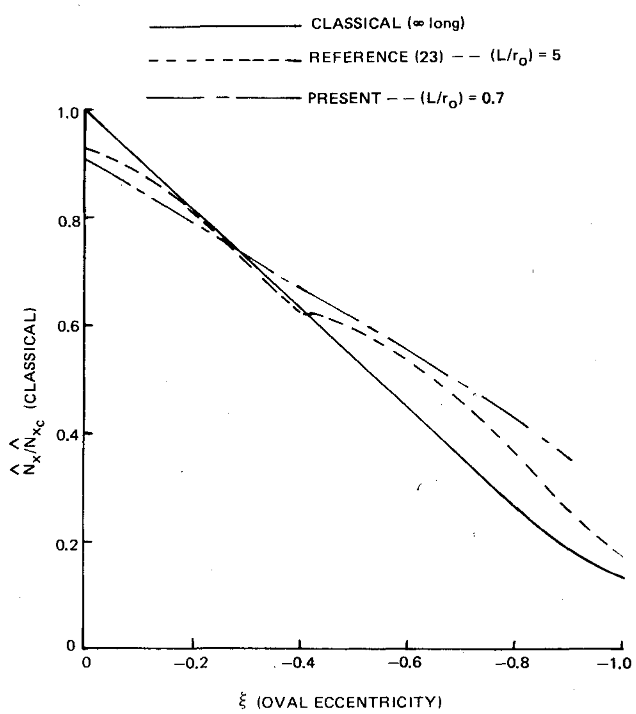


Fig. 5 Comparison of axial compressive buckling load of finite, unstiffened, oval shell with results of Ref. 23.

large differences in load λ_l for the circular case of the linearized approach; that is, λ_l is not approximately equal to 1.

Presented in Table 4 are typical eigenvectors obtained when eigenvalues of Eq. (20) are determined. As it can be observed, the externally stiffened case shows that the extreme (largest and lowest value of n) modes contribute little to the expansion whereas for the internally stiffened problem they are just as important as the others. This is indicative of the sufficiency of the number of modes considered for one case

Table 4 Eigenvectors of externally and internally stiffened, finite, simple supported, oval cylinders under axial compression, S2 case

	Externally stiffened cylinder ($\xi = -0.2$)[a] Eigenvector	
X_n	Y_n	Z_n
−0.228904550 E−02	0.730386089 E−03	−0.644866386 E−02
0.906725397 E−02	0.272238461 E−02	0.890772908 E−01
0.322003817 E−01	0.985377807 E−01	0.637491272 E+00
0.425790574 E−01	0.142388341 E+00	0.100000000 E+01
0.384601120 E−02	−0.105137941 E−02	0.864209026 E−01
0.619161396 E−04	0.534608868 E−04	0.436499172 E−03
0.309650654 E−05	−0.401086588 E−06	0.415655405 E−04
0.676229181 E−06	−0.260707590 E−06	−0.110145936 E−06
−0.358389839 E−07	0.240149359 E−07	−0.520395734 E−07

	Internally stiffened cylinder ($\xi = -0.2$)[b] Eigenvector	
X_n	Y_n	Z_n
0.864404378 E−02	0.847922116 E−01	0.950131849 E+00
0.872658756 E−02	0.847921110 E−01	0.982594870 E+00
0.860156874 E−02	0.847927132 E−01	0.992522836 E+00
0.843439736 E−02	0.847935465 E−01	0.997729982 E+00
0.824315718 E−02	0.847945279 E−01	0.100000000 E+01
0.803198797 E−02	0.847956233 E−01	0.999580024 E+00
0.780206437 E−02	0.847968121 E−01	0.996373463 E+00
0.754118350 E−02	0.847981323 E−01	0.988529648 E+00
0.710639091 E−02	0.848000211 E−01	0.956412295 E+00

[a] Eigenvalue = 0.913807 E + 00. [b] Eigenvalue = 0.972298 E + 00.

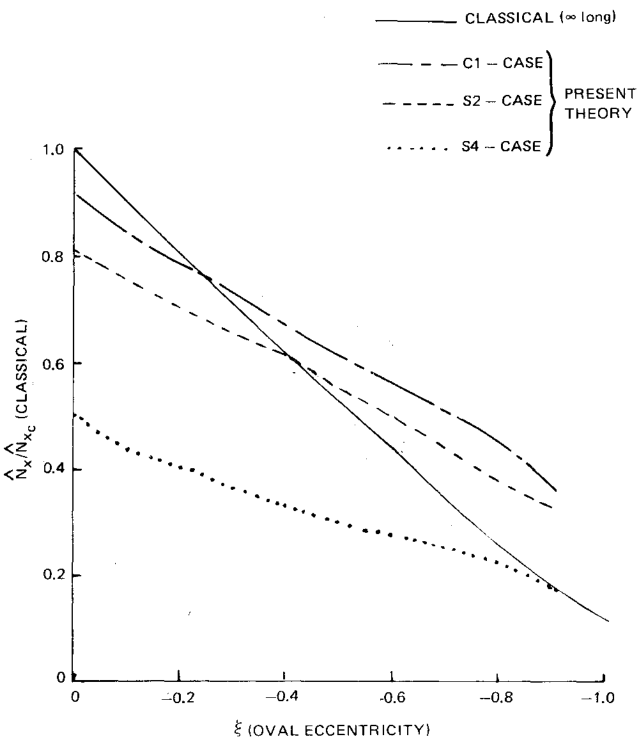


Fig. 6 Variation of the axial compressive buckling load of finite, unstiffened, oval cylinder with boundary conditions C1, S2, and S4.

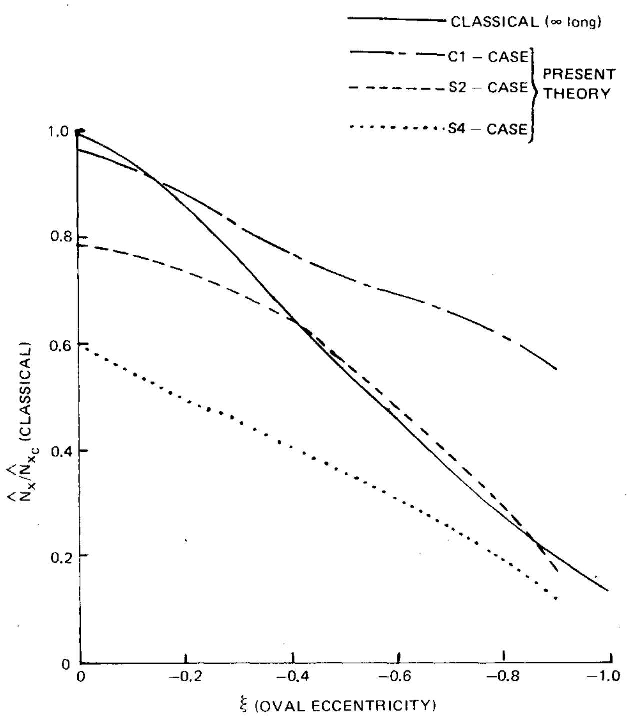


Fig. 7 Variation of the axial compressive buckling load of finite, externally stiffened, oval cylinder with boundary conditions C1, S2, and S4.

(externally stiffened) and the inadequacy for the other (internally stiffened).

An attempt to solve the internally stiffened problem was made by including up to twenty modes in the expansion. This resulted in much larger matrices (60×60 vs 27×27) and use of over twenty times more computer time, but to no avail. The reason is that for this particular problem all of the modes included have almost the identical shape. This means that in effect only one mode was used. With good probability this problem could be solved by considering modes that have $(n+6)$ or $(n+8)$ instead of $(n+2)$ number of waves from each other. But, this is merely a matter of conjecture and it was not done.

The results for the axial compression problems of the unstiffened and externally stiffened problems are summarized in Figs. 6 and 7. The curves for the unstiffened cylinder are those of the linearized approach, while those of the stiffened cylinder are those employing the engineering approximation for the nonlinear terms.

In conclusion, the eigenfunction expansion approach for the problems in this study proved to be successful and more advantageous than other approaches (i.e., Ref. 23). The application of the method is very general, with results obtained here comparing quite favorably with those of previously published solutions.

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